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MAXIMUM AND k-TH MAXIMAL SPANNING TREES OF A WEIGHTED GRAPH

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Let A be a maximum spanning tree and P be an arbitrary spanning tree of a connected weighted graph G. Then we prove that there exists a bijection ψ from $A \setminus P$ into $P \setminus A$ such that for any edge $a \in A \setminus P$, $(P \setminus \psi(a)) \cup a$ is a spanning tree of G whose weight is greater than or equal to that of P. We apply this theorem to some problems concerning spanning trees of a weighted graph.

1. Introduction

Let G be a connected graph with edge set $E(G)$, which may have multiple edges but has no loops. With each edge e of G, a real number $W(e)$, called its weight, is associated. The weight $W(P)$ of a spanning tree P of G is the sum of the weights of all the edges in P. We denote all the weights of spanning trees of G by $W_1 > W_2 > \dots$ $\dots > W_m$. A spanning tree of G with weight W_k is called a *k-th maximal spanning tree* of G. A first maximal spanning tree and an m-th one are usually called a *maximum spanning tree* and a *minimum spanning tree*, respectively. We denote by Ω_k the set of k-th maximal spanning trees of G, and by Ω the set of all the spanning trees of G. Then, Ω is classified into disjoint union of $\{\Omega_k\}$ as follows: $\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_m$. This classification is important in electrical network theory (for instance, see [5]). In this paper we shall give some results on Ω_k and k-th maximal spanning trees.

In order to describe our results, we define some notation. For convenience, we call a cycle a *tieset*, and consider a spanning tree, a tieset and a cutset of G as subsets of $E(G)$. Let P and Q be spanning trees of G. Then, $P \setminus Q$ denotes the set of edges which are in P but not in Q, and the *distance* $d(P, Q)$ between P and Q is defined to be the number of edges in $P \setminus Q$ (i.e. $d(P, Q) = |P \setminus Q| = |Q \setminus P|$). We write $L_r(P)$ for the set of spanning trees R of G such that $d(P, R) \leqq r$ for an integer r.

Denote by $\bar{P}$ the complement of P, that is, $\bar{P} = E(G) \setminus P$. Let x and y be edges of G such that $x \in P$ and $y \notin P$. Then, $P \cup y$ includes a unique tieset $T(P \cup y)$ (i.e. the fundamental cycle of G defined by y with respect to P) and $\bar{P} \cup x$ includes a unique cutset $C(\bar{P} \cup x)$ of G (i.e. the fundamental cutset of G defined by x with respect to $\bar{P}$). For example, let $P = \{a, p, q, r, s, t\}$ be a spanning tree of the weighted graph given in Figure 1. Then $T(P \cup b) = \{b, t, r, q, p, a\}$ and $C(\bar{P} \cup r) = \{r, e, d, b\}$. We now give our main theorem.

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Theorem 1. *Let A be a maximum spanning tree and P be an arbitrary spanning tree of a connected weighted graph G. Then there exists a bijection ψ from $A \setminus P$ into $P \setminus A$ which satisfies the following condition for every edge $a \in A \setminus P$.*

(1.1) $\quad Q = (P \setminus \psi(a)) \cup a$ *is a spanning tree of G and $W(Q) \geqq W(P)$.*

Note that the condition (1.1) is equivalent to

(1.2) $$\psi(a) \in T(P \cup a) \quad \text{and} \quad W(\psi(a)) \leqq W(a).$$

Moreover, if A is a minimum spanning tree of G, then the theorem also holds by inverting the inequalities in (1.1) and (1.2). We now give an example. Let $A = \{a, b, c, d, e, f\}$ and $P = \{a, p, q, r, s, t\}$ be spanning trees of the weighted graph in Figure 1. Then A is a maximum spanning tree and a bijection ψ defined by $\psi(b) = r$, $\psi(c) = p$, $\psi(d) = q$, $\psi(e) = s$ and $\psi(f) = t$ satifies the conditions (1.1) and (1.2).

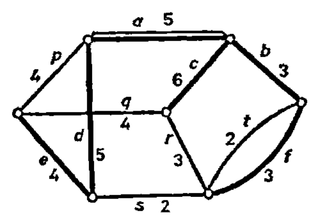

Fig. 1. A weighted graph and two spanning trees. Number denotes the weight of each edge

We shall apply Theorem 1 to some problems concerning k-th maximal spanning trees and Ω_k. In particular, we prove the following conjecture for some k's by making use of Theorem 1.

Conjecture. Let G be a connected weighted graph. Then the following statements hold for every k:

(α) If A is a maximum spanning tree of G, then $L_{k-1}(A)$ contains an i-th maximal spanning tree of G for every i, $1 \leqq i \leqq k$.

(β) If P is a k-th maximal spanning tree in $L_k(P)$, then P is a k-th maximal spanning tree of G.

(γ) Let $\Gamma(\Omega_k, l)$ denote the graph whose vertex set is Ω_k and whose edge set is the set of unordered pairs $\{P, Q\}$ such that $P, Q \in \Omega_k$ and $d(P, Q) \leqq l$. Then $\Gamma(\Omega_k, k)$ is connected.

(δ) If P is a k-th maximal spanning tree of G, then $L_{k-1}(P)$ contains an i-th maximal spanning tree of G for every i, $1 \leqq i \leqq k$.

It is well-known that (β) and (γ) hold if $k=1$ ([1, p. 61], [2, p. 175], Kruskal [7], Prim [9]). Kawamoto, Kajitani and Shinoda [6] proved that (α), (β) and (γ) hold if $k=2$, and (δ) follows if $k=3$, but their proofs are complicated and long. We shall give simple proofs to the above known results by using Theorem 1, and prove that (α) holds if $k=3$ or 4, (β) and (γ) hold if $k=3$, and (δ) follows if $k=4$.

Note that if the conjecture (α), (β), (γ) and (δ) follow, then they are best possible in the sense that there exists a connected weighted graph G in which; (α) for a maximum spanning tree A of G, $L_{k-2}(A)$ does not contain any k-th maximal spanning trees of G; (β) a spanning tree P of G is not a k-th maximal spanning tree of G even if P is a k-th maximal spanning tree in $L_{k-1}(P)$; (γ) $\Gamma(\Omega_k, k-1)$ is not connected; or (δ) for a k-th maximal spanning tree P of G, $L_{k-2}(P)$ does not contain any i-th maximal spanning trees of G for some i, $1 \leqq i \leqq k$. Such a weighted graph can be obtained from a graph given in Figure 2 by assigning a suitable weight to each edge. In particular, some theorems on the conjecture are best possible in the above sense.

Fig. 2

2. Proof of Theorem 1 and preliminaries

We begin with the next easy lemma, which will be used without any mention.

Lemma 1. *Let P be a spanning tree of a connected graph G, and x and y be edges of G such that $x \in P$ and $y \notin P$. Then the following three statements are equivalent:*

(1) $(P \setminus x) \cup y$ *is a spanning tree of* G.
(2) $x \in T(P \cup y)$.
(3) $y \in C(\bar{P} \cup x)$.

Proof of Theorem 1. By Lemma 1, it is sufficient to show that there exists a bijection ψ from $A \setminus P$ into $P \setminus A$ which satisfies the condition (1.2). Let $A \setminus P = \{a_1, \ldots, a_n\}$. For each a_i, put

$$T_i = T(P \cup a_i) = a_i \cup A_i \cup Q_i \cup R_i, \tag{2.1}$$

where $A_i = \{x \in T_i | x \in A \cap P\}$, $Q_i = \{x \in T_i | x \in P \setminus A, \quad W(x) > W(a_i)\}$ and $R_i = \{x \in T_i | x \in P \setminus A, W(x) \leqq W(a_i)\}$. Then, every T_i is a disjoint union of a_i, A_i, Q_i and R_i. We first prove that for any k $(1 \leqq k \leqq n)$ elements $R_{i1}, \ldots, R_{ik}$ in $\{R_1, \ldots, R_n\}$, it follows that

$$|R_{i1} \cup \ldots \cup R_{ik}| \geqq k. \tag{2.2}$$

In order to prove (2.2), we assume that there exists $R_1, \ldots, R_k$ such that $|R_1 \cup \ldots \cup R_k| < k$, and derive a contradiction. Consider a vector space

$$V = \left\{\sum \alpha_i e_i \,|\, \alpha_i \in \mathrm{GF}(2) \text{ and } e_i \in E(G)\right\}$$

over $\mathrm{GF}(2) = \{0, 1\}$ generated by $E(G)$. Since $|R_1 \cup \ldots \cup R_k| < k$, $R_1, \ldots, R_k$ are linearly dependent in V. Then there exist elements $\beta_1, \ldots, \beta_k$ of GF(2) such that $\sum \beta_i R_i = 0$ and some $\beta_t \neq 0$. Hence we have by (2.1) that

$$\sum \beta_i T_i = \sum \beta_i a_i + \sum \beta_i A_i + \sum \beta_i Q_i,$$

where $\sum \beta_i T_i$ forms a disjoint union of tiesets. Let a' be an edge of minimum weight in the set of edges a_i whose coefficient $\beta_i \neq 0$. Since a' is contained neither in A_j nor in Q_j $(1 \leqq j \leqq k)$, there exists a tieset T' that contains a' in the set of tiesets whose disjoint union is $\sum \beta_i T_i$. We write $T' = a' \cup H \cup A' \cup Q'$, where

$$H = T' \cap (\sum \beta_i a_i \setminus a'),\ A' = T' \cap (\sum \beta_i A_i) \ \text{ and } \ Q' = T' \cap (\sum \beta_i Q_i) \subset P \setminus A.$$

Since $\emptyset \neq (T' \cap C(\bar{A} \cup a')) \setminus a' = Q' \cap C(\bar{A} \cup a')$, there exists an edge p' in $Q' \cap C(\bar{A} \cup a')$. By our choice of p', there is an integer t such that $\beta_t \neq 0$ and $p' \in Q_t$, and so $W(p') > W(a_t) \geqq W(a')$. Hence $(A \setminus a') \cup p'$ is a spanning tree of G whose weight is greater than that of A, which establishes the required contradiction. Consequently, (2.2) holds.

Therefore, by Hall's marriage theorem [4], $\{R_i | 1 \leqq i \leqq n\}$ has a system of distinct representatives $\{p_i | p_i \in R_i,\ 1 \leqq i \leqq n\}$. It is obvious that the bijection ψ defined by $\psi(a_i) = p_i$ for all i, $1 \leqq i \leqq n$, satisfies the desired condition (2.2). Consequently, the theorem is proved. ∎

We first apply Theorem 1 to prove the following theorem which characterizes maximum spanning trees of a weighted graph.

Theorem 2. ([1, p. 61], [2, p. 175], [7], [9]) *Let A be a spanning tree of a connected weighted graph G. Then the following statements are equivalent.*

(1) *A is a maximum spanning tree of G.*
(2) *A is a maximum spanning tree in $L_1(A)$.*
(3) *$W(a) \geqq W(x)$ for every $x \in E(G) \setminus A$ and $a \in T(A \cup x) \setminus x$.*

Proof. It is easy to see that (2) and (3) are equivalent by Lemma 1, and the fact that (1) implies (2) is trivial. So it suffices to show that (2) implies (1). Let P be a maximum spanning tree of G and A be a spanning tree which satisfies (2). By Theorem 1, we can write $P \setminus A = \{p_1, \ldots, p_n\}$ and $A \setminus P = \{a_1, \ldots, a_n\}$ so that $Q_i = (A \setminus a_i) \cup p_i$ is a spanning tree of G and $W(Q_i) \geqq W(A)$ for every i. Since $Q_i \in L_1(A)$, we have $W(Q_i) = W(A)$ by (2), and conclude $W(a_i) = W(p_i)$. Hence $W(A) = W(P)$. Consequently, (2) implies (1). ∎

Lemma 2. *Let P and Q be spanning trees of a connected graph. Then there exists a bijection ψ from $P \setminus Q$ into $Q \setminus P$ such that $\psi(p) \in T(Q \cup p)$ for every $p \in P \setminus Q$.*

Proof. Let $P \setminus Q = \{p_1, \ldots, p_n\}$, and put $T_i = T(Q \cup p_i) = p_i \cup P_i \cup Q_i$, where $P_i = \{x \in T_i | x \in P \cap Q\}$ and $Q_i = \{x \in T_i | x \in Q \setminus P\}$. Then $\{Q_i | 1 \leqq i \leqq n\}$ satisfies the condition of Hall's marriage theorem [4], and thus the lemma holds. ∎

Lemma 3. *Let P and Q be spanning trees of a connected weighted graph. If $W(P) > W(Q)$, then there exist $p \in P \setminus Q$ and $q \in Q \setminus P$ such that $W(p) > W(q)$ and $q \in T(Q \cup p)$.*

Proof. This is an easy consequence of Lemma 2. ∎

Lemma 4. *Let P be a spanning tree of a connected graph G, and let x and y be two edges of G not contained in P, and p and q be two edges of P such that $p \in T(P \cup x)$ and $q \in T(P \cup y)$. If $p \notin T(P \cup y)$ or $q \notin T(P \cup y)$, then $(P \setminus \{p, q\}) \cup \{x, y\}$ is a spanning tree of G.*

Proof. Suppose $p \notin T(P \cup y)$. It is immediate that $R = (P \setminus p) \cup x$ is a spanning tree and $q \in T(R \cup y) = T(P \cup y)$. Therefore $(R \setminus q) \cup y = (P \setminus \{p, q\}) \cup \{x, y\}$ is a spanning tree. ∎

Lemma 5. *Let P and Q be spanning trees of a connected graph G such that* $d(P, Q) \geqq 2$. *Put* $P \setminus Q = \{p_1, \ldots, p_n\}$ *and* $Q \setminus P = \{q_1, \ldots, q_n\}$ *such that* $q_i \in T(Q \cup p_i)$ *for every i* (Lemma 2). *Then there exist i and j for which* $(Q \setminus \{q_i, q_j\}) \cup \{p_i, p_j\}$ *is a spanning tree of G.*

Proof. Suppose $T(Q \cup p_1) \supset Q \setminus P$ and $T(Q \cup p_2) \supset Q \setminus P$. Then the symmetric difference $T(Q \cup p_1) + T(Q \cup p_2)$ (mod 2) is a set of tiesets and included in a spanning tree P, a contradiction. Hence $T(Q \cup p_i) \not\supset Q \setminus P$ for some $i \in \{1, 2\}$. Take $q_j \in (Q \setminus P) \setminus T(Q \cup p_i)$. Then $(Q \setminus \{q_i, q_j\}) \cup \{p_i, p_j\}$ is a spanning tree by Lemma 4. ∎

3. Special weighted graphs and applications of Theorem 1

In this section we shall deal with weighted graphs in which the sequence $\{W_n\}$ of weights of spanning trees is an arithmetic progression, that is, the difference $W_{n+1} - W_n$ is constant for all n. A weighted graph in which the weight of each edge is one of $\{-1, 0, 1\}$ corresponds to an electrical network, and so such a special weighted graph is important in applications. We first give a short proof to the following theorem by using Theorem 1.

Theorem 3. (Hakimi and Maeda [3]) *Let G be a connected weighted graph such that the set of edge-weights is* $\{w, w+d, w+2d\}$ $(d > 0)$. *Then the sequence* $\{W_n\}$ *of G is an arithmetic progression.*

Proof. Let P be a k-th maximal spanning tree and Q be a $(k+1)$-th maximal spanning tree of G. Then, by Lemma 3, we can take $p \in P \setminus Q$ and $q \in Q \setminus P$ such that $q \in T(Q \cup p)$ and $W(q) < W(p)$. Put $R = (Q \setminus q) \cup p$. Then R is a spanning tree and $W(P) - W(Q) \leqq W(R) - W(Q) = W(p) - W(q) \leqq w + 2d - w = 2d$. Therefore $W_k - W_{k+1} = d$ or $2d$ for all k. We consider two cases.

Case 1. $W_1 - W_2 = 2d$. We assume that $\{W_1, \ldots, W_k\}$ is an arithmetic progression with difference $-2d$ but $W_{k+1} - W_k = -d$ $(k \geqq 2)$, and derive a contradiction. Let A be a maximum spanning tree, and choose a $(k+1)$-th maximal spanning tree P so that $d(A, P)$ is minimum. By Theorem 1 and by our choice of P, we can write $A \setminus P = \{a_1, \ldots, a_n\}$ and $P \setminus A = \{p_1, \ldots, p_n\}$ such that $p_i \in T(P \cup a_i)$ and $W(p_i) < W(a_i)$. If $W(a_j) - W(p_j) = 2d$ for some j, then $R = (P \setminus p_j) \cup a_j$ is a spanning tree and $W_{k-1} > W(R) > W_k$, a contradiction. Hence $W(a_i) - W(p_i) = d$ for all i. By Lemma 5, we can take i and j so that $S = (P \setminus \{p_i, p_j\}) \cup \{a_i, a_j\}$ is a spanning tree of G. Since $W_{k-1} > W(S) > W_k$, we have a desired contradiction.

Case 2. $W_1 - W_2 = d$. Suppose $\{W_1, \ldots, W_k\}$ is an arithmetic progression with difference $-d$ but $W_{k+1} - W_k = -2d$ $(k \geqq 2)$. Let A be a maximum spanning tree and P be a $(k+1)$-th maximal spanning tree. Then we can set $A \setminus P = \{a_1, \ldots, a_n\}$ and $P \setminus A = \{p_1, \ldots, p_n\}$ so that $p_i \in T(P \cup a_i)$ and $W(p_i) \leqq W(a_i)$ by Theorem 1. If $W(a_j) - W(p_j) = d$ for some j, then we have a contradiction considering $(P \setminus p_j) \cup a_j$. Hence $W(a_i) - W(p_i) = 0$ or $2d$ for every i, and so $W_1 - W_{k+1} = 2dl$

for some integer l. Let B be a second maximal spanning tree, and choose a $(k+1)$-th maximal spanning tree Q so that $d(B, Q)$ is minimum. By Lemma 2, we can write $B\setminus Q=\{b_1, ..., b_r\}$ and $Q\setminus B=\{q_1, ..., q_r\}$ with $q_i\in T(Q\cup b_i)$ for every i. By the same argument as above, we have $W(b_j)-W(q_j)=2d$ if $W(b_j)>W(q_j)$. Since $W_2-W_{k+1}=2dl-d$ for the integer l, we may assume $W(b_1)-W(q_1)=2d$ and $W(b_2)-W(q_2)=-d$. If $q_2\in T(Q\cup b_1)$, then $W(b_1)=w+2d$ and $W(q_2)=w+d$ since otherwise we can obtain a $(k+1)$-th maximal spanning tree Q' with $d(B, Q')=$ $=d(B, Q)-1$, which contradicts the choice of Q. Thus $W_k>W((Q\setminus q_2)\cup b_1)>W_{k+1}$, a contradiction. Therefore $q_2\notin T(Q\cup b_1)$. In this case it follows from Lemma 4 that $R=(Q\setminus\{q_1, q_2\})\cup\{b_1, b_2\}$ is a spanning tree. Since $W_k>W(R)>W_{k+1}$, we obtain a required contradiction. ▮

The next theorem says that the conjecture (α) is true if $\{W_n\}$ is an arithmetic progression.

Theorem 4. *Let G be a connected graph whose $\{W_n\}$ is an arithmetic progression, and A be a maximum spanning tree of G. Then for any integer k $(k\geqq 1)$, $L_{k-1}(A)$ contains an i-th maximal spanning tree of G for all i, $1\leqq i\leqq k$.*

Proof. For any t, $1\leqq t\leqq k$, choose a t-th maximal spanning tree P of G so that the distance $d(A, P)$ is minimum. Suppose $d(A, P)\geqq k$. By Theorem 1, let $A\setminus P=$ $=\{a_1, ..., a_n\}$ and $P\setminus A=\{p_1, ..., p_n\}$ such that $p_i\in T(P\cup a_i)$ and $W(p_i)\leqq W(a_i)$ for every i, where $n=d(A, P)\geqq k$. Then it follows from the choice of P that $W(a_i)>$ $>W(p_i)$ for every i. Choose $a'=a_j$ and $p'=p_j$ in such a way that $W(a_j)-W(p_j)$ is minimum, and put $R=(P\setminus p')\cup a'$. Since $\{W_n\}$ is an arithmetic progression, we have

$$(3.1)\qquad W(A)-W(P) = W_1-W_t \leqq W_1-W_k = (k-1)(W_{t-1}-W_t) \leqq$$
$$\leqq (k-1)\big(W(R)-W(P)\big) = (k-1)\big(W(a')-W(p')\big).$$

On the other hand,

$$W(A)-W(P) = \sum\big(W(a_i)-W(p_i)\big) \geqq n\big(W(a')-W(p')\big) \geqq k\big(W(a')-W(p')\big).$$

This contradicts (3.1), and we conclude that the theorem follows. ▮

We denote the weights of all the edges of a weighted graph G by $w_1>w_2>...$ $...>w_l$. Note that if G is 2-connected, then $W_1-W_m\geqq w_1-w_l$ since G has a tieset which contains any two given edges of G. We prove the following theorem by using Theorem 1.

Theorem 5. (Okamoto and Kajitani [8]) *Let G be a connected weighted graph with $W_1-W_m\geqq w_1-w_l$. Let α and β be real numbers such that $W_m\leqq\beta<\alpha\leqq W_1$ and $\alpha-\beta\geqq w_1-w_l$. Then*

$$E(G) = \cup P,$$

where the union is over all spanning trees P of G such that $\beta\leqq W(P)\leqq\alpha$.

Proof. Put $F=\cup P$, which is defined in the theorem. Suppose $E(G)\neq F$. Let $e\in E(G)\setminus F$ and P be a spanning tree containing e. Then $W(P)<\beta$ or $\alpha<W(P)$. We first assume $W(P)<\beta$. Choose a spanning tree Q containing e in such a way that $W(Q)$ is as large as possible subject to $W(Q)<\beta$. Let A be a maximum spanning tree. If $e\in A\cap Q$, then there are edges $a\in A\setminus Q$ and $q\in Q\setminus A$ such that $q\in T(Q\cup a)$ and $W(q)<W(a)$ by Lemma 3. Then $R=(Q\setminus q)\cup a$ is a spanning tree containing e, and $W(Q)<W(R)$, which implies $\beta\leqq W(R)$ by the choice of Q. Moreover, $W(R)\leqq W(Q)+w_1-w_l\leqq W(Q)+\alpha-\beta\leqq\alpha$. Hence $e\in F$ as $e\in R$, a contradiction.

Therefore $e\in Q\setminus A$. By Theorem 1, we write $A\setminus Q=\{a_1, ..., a_n\}$ and $Q\setminus A=\{q_1=e, q_2, ..., q_n\}$ such that $R_i=(Q\setminus q_i)\cup a_i$ is a spanning tree and $W(Q)\leqq W(R_i)$ for every i. If $W(R_j)>W(Q)$ for some j, $2\leqq j\leqq n$, then we obtain $e\in F$ by the same argument on $R=(Q\setminus q)\cup a$. Hence $W(R_i)=W(Q)$, that is, $W(a_i)=W(q_i)$ for all i, $2\leqq i\leqq n$. Then we have $W(A)-W(Q)=W(a_1)-W(q_1)\leqq w_1-w_l$. On the other hand, $W(A)-W(Q)>\alpha-\beta\geqq w_1-w_l$, a contradiction. If $W(P)>\alpha$, then we can also derive a contradiction by using Theorem 1 on minimum spanning trees. Consequently, the proof is complete. ▮

Corollary 1. *Let G be a connected weighted graph whose edge-weight-set is $\{w, w+d, w+2d\}$ $(d>0)$. If $W_1>W_k>W_{k+1}>W_m$, then $E(G)=\{e\mid$ There exists a spanning tree P such that $e\in P$ and $W_k\geqq W(P)\geqq W_{k+1}\}$.*

Proof. By Theorem 3, $\{W_n\}$ is an arithmetic progression with difference $-d$ or $-2d$. Put $\alpha=W_k+d/2\leqq W_1$ and $\beta=W_{k+1}-d/2\geqq W_m$. Then the corollary follows at once from Theorem 5. ▮

4. Maximum and k-th maximal spanning trees

Kawamoto, Kajitani and Shinoda [6] proved the following theorem in the case of $k=2$. But its proof is not so easy. We give a simple proof of it together with some new results.

Theorem 6. *Let G be a connected weighted graph and A be a maximum spanning tree of G. If $k\in\{1, 2, 3, 4\}$, then $L_{k-1}(A)$ contains an i-th maximal spanning tree of G for every i, $1\leqq i\leqq k$.*

We prove the next lemma instead of the above theorem because the theorem is an easy consequence of the lemma.

Lemma 6. *Let G, A and k be the same as in Theorem 6, and P be a k-th maximal spanning tree of G. If $d(A, P)\geqq k$, then there exists a k-th maximal spanning tree P' such that $d(A, P')=d(A, P)-1$ and $d(P, P')=1$.*

Proof. Suppose $d(A, P)\geqq k$. It suffices to show that there exist $a\in A\setminus P$ and $p\in P\setminus A$ such that $a\in T(P\cup a)$ and $W(p)=W(a)$. We consider four cases.

Case 1. $k=1$. It follows immediately from Theorem 1.

Case 2. $k=2$. Let B be a second maximal spanning tree of G such that $d(A, B)\geqq 2$. By Theorem 1, we can write $A\setminus B=\{a_1, ..., a_n\}$ and $B\setminus A=\{b_1, ..., b_n\}$ so that $W(a_i)\geqq W(b_i)$ and $b_i\in T(B\cup a_i)$ for every i. We may assume $W(a_1)>W(b_1)$. Then $(B\setminus b_1)\cup a_1$ is a maximum spanning tree, and so $W(a_i)=W(b_i)$ for all i, $i\neq 1$.

Case 3. $k=3$. Let C be a third maximal spanning tree of G such that $d(A,C)\geqq 3$. By Theorem 1, we can write $A\setminus C=\{a_1,\ldots,a_n\}$ and $C\setminus A=\{c_1,\ldots,c_n\}$ so that $W(a_i)\geqq W(c_i)$ and $c_i\in T(C\cup a_i)$. If $(C\setminus c_j)\cup a_j$ is a maximum spanning tree for some j, then we have $W(a_i)=W(c_i)$ for all i, $i\neq j$. Hence we may assume that $(C\setminus c_i)\cup a_i$ is a second maximal spanning tree of G for all i. By Lemma 5, we can take s and t such that $A'=(C\setminus\{c_s,c_t\})\cup\{a_s,a_t\}$ is a spanning tree of G. Since $W(A')>W((C\setminus c_t)\cup a_t)$, A' is a maximum spanning tree of G, and so $W(c_i)=W(a_i)$ for every i, $i\notin\{s,t\}$, a contradiction.

Case 4. $k=4$. We omit the proof of this case since it is long and similar to that of Case 3. ∎

The next theorem can be proved similarly as Theorem 8, so we omit its proof.

Theorem 7. (Kawamoto, Kajitani and Shinoda [6]) *Let G be a connected weighted graph. Then the following statements hold.*

(1) *If B is a second maximal spanning tree in $L_2(B)$, then B is a second maximal spanning tree of G.*

(2) *The graph $\Gamma(\Omega_2,2)$ is connected (see conjecture (γ)).*

(3) *If C is a third maximal spanning tree of G, then $L_2(C)$ contains a first, a second and a third maximal spanning tree of G.* ∎

Lemma 7. *Let B and C be a second and a third maximal spanning tree of a connected weighted graph G, respectively. Then*

(1) *If $d(B,C)\geqq 4$, then G has a third maximal spanning tree C' such that $d(B,C')=d(B,C)-1$ and $d(C,C')=1$.*

(2) *If $d(B,C)=3$, then G has a third maximal spanning tree C'' such that $d(B,C'')\leqq 2$ and $d(C,C'')\leqq 2$,*

Proof. We prove only (1) because the proof of (2) is similar to that of (1) and long. We prove (1) by induction on $|E(G)|$. Suppose $d(B,C)\geqq 4$. If $B\cup C\neq E(G)$, then we get a desired spanning tree by applying the inductive hypothesis or Lemma 6 to the graph, in which B and C are a second and a third maximal spanning trees or a first and a second maximal spanning trees, obtained from G by deleting the edges in $E(G)\setminus(B\cup C)$. If $B\cap C\neq\emptyset$, then we also have a desired spanning tree considering the graph obtained from G by contracting the edges in $B\cap C$. Hence we may assume $E(G)=B\cup C$ and $B\cap C=\emptyset$. By Theorem 2, there is a maximum spanning tree A with $d(A,B)=1$. Since $d(A,C)\geqq d(B,C)-d(A,B)\geqq 3$, G has a third maximal spanning tree C' such that $d(A,C')=d(A,C)-1$ and $d(C,C')=1$ by Lemma 6. Since $E(G)$ is a disjoint union of B and C, we have $d(B,C')=d(B,C)-1$. Consequently, the proof is complete. ∎

Lemma 8. *Let B be a second maximal spanning tree of a connected weighted graph G. Then $L_2(B)$ contains a first, a second and a third maximal spanning tree of G.*

Proof. This lemma is an easy consequence of Theorem 2 and Lemma 7. ∎

Theorem 8. *Let G be a connected weighted graph. Then the following statements hold.*

(1) *If C is a third maximal spanning tree in $L_3(C)$, then C is a third maximal spanning tree of G.*

(2) *The graph $\Gamma(\Omega_3, 3)$ is connected (see conjecture (γ)).*

(3) *If D is a fourth maximal spanning tree of G, then $L_3(D)$ contains a first, a second, a third and a fourth maximal spanning tree of G.*

Proof.. (1). Suppose $L_1(C)$ contains a first or a second maximal spanning tree of G. Then $L_3(C)$ contains a first, a second and a third maximal spanning tree of G by Theorem 6 or Lemma 8. Hence C is a third maximal spanning tree of G. So we may assume that $L_1(C)$ contains neither a first nor a second maximal spanning tree of G. Then, by Theorem 2, G has spanning trees P and Q such that $P \in L_1(C)$, $Q \in L_1(P)$ and $W(C) < W(P) < W(Q)$. If Q is a maximum spanning tree of G, then $L_1(Q)$ contains a second maximal spanning tree B of G, and so $L_3(C)$ contains P, Q and B, which is contrary to the assumption of (1). If Q is not a maximum spanning tree of G, then we can obtain a contradiction by using Theorem 2. Hence (1) is proved.

(2). Let C and C' be any two distinct third maximal spanning trees of G such that $d(C, C') \geqq 4$. By Theorem 2, $L_1(C)$ contains a first or a second maximal spanning tree of G. First suppose $L_1(C)$ contains a maximum spanning tree A of G. Then by Lemma 6, there exists a sequence $\{C_1 = C', C_2, \ldots, C_r\}$ of third maximal spanning trees of G such that $d(C_i, C_{i+1}) = 1$, $1 \leqq i \leqq r-1$, and $d(C_r, A) = 2$. Since $d(C, C_r) \leqq d(C, A) + d(A, C_r) = 3$, C and C' are contained in the same component of $\Gamma(\Omega_3, 3)$. If $L_1(C)$ contains a second maximal spanning tree of G, then we can prove similarly by using Lemma 7 that C and C' are contained in the same component of $\Gamma(\Omega_3, 3)$. Consequently, $\Gamma(\Omega_3, 3)$ is connected.

(3) It suffices to show that D is a fourth maximal spanning tree in $L_3(D)$. If D is an i-th maximal spanning tree in $L_3(D)$ for some $i \in \{1, 2, 3\}$, then D is an i-th maximal spanning tree of G by Theorems 2 and 7 and (1) of this theorem. Hence we have a contradiction, and conclude that D is a fourth maximal spanning tree in $L_3(D)$. ∎

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